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Variable Structure Controller Design for Linear Systems with Bounded Inputs

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Abstract: This paper studies the design of variable structure systems with saturation inputs. Sliding mode domain, reaching domain, and unescapable reaching domain of linear systems with variable structure are defined and investigated. When the state matrix of the linear system is Hurwitz, the stability of the variable structure systems is proven by using passivity theory. Moreover, variable structure systems with novel nonlinear switching surfaces are proposed for second order systems. Two strategies for designing variable structure control for high order linear systems are also proposed, such as step-by-step variable structure control and moving-surface variable structure control, which were found to guarantee that the reaching condition of the variable structure control is always satisfied. Finally, an illustrative example pertaining to the attitude control of a flexible spacecraft demonstrates the effectiveness of the proposed methods.

Keywords: Bounded inputs, linear systems, nonlinear switching surface, passivity theory, variable structure control.

1. INTRODUCTION

Variable Structure Control (VSC) with sliding mode was first proposed in the early 1950's. Nowadays, VSC has developed into a general design method that is being examined for a wide spectrum of system types including nonlinear systems, MIMO systems, discrete-time models, large-scale and infinite-dimensional systems, and stochastic systems. The published research provided by Utkin [1], DeCarlo [2], Hung [3] and Young [4] has presented the fundamental theory and design methods of VSC in different aspects. The most distinguished feature of VSC is claimed to result in insensitivity to parameter variations, and complete rejection of disturbances. This much desirable system performance only holds in the sliding mode domain (SMD) on the switching surfaces, which is easily satisfied under nonsaturating control.

An important problem encountered in practice is that of control input saturation, which originates from actuators in system realization. For example, it is of particular interest in spacecraft control where the control objectives are to be achieved with limited control authority [5]. When the control input is bounded, the SMD will be restricted to some local domain near zero on the switching surface. The motion outside of the SMD is so-called 'bang-bang' motion, which does not have robustness. Thus, restrictions of control input can lead to substantial performance deterioration and even to instability of the entire system. Stability analysis of linear systems with actuators having amplitude saturation has been a basic problem in the literature [6]. However, studies on VSC from such a point of view are rare [7-9]. The work done by Madani-Esfahani *et al*. [7] investigated regions of asymptotic stability of uncertain variable structure control systems with bounded controllers. The work performed by Okabayashi *et al*. [8] investigated the design of VSC for LTI systems and presented a new method of designing nonlinear switching surfaces for second order LTI systems. The work performed by Han et al. [9] added a nonlinear part to the traditional linear sliding surface to ensure that the control signal generated by the controller does not exceed the bounds of the system input. In this paper, we aim to find an effective design methodology for linear systems with bounded inputs.

The main contributions of this paper are summarized as follows: (a) the concepts of reaching domain (RD) and unescapable reaching domain (URD) are introduced to investigate variable structure controller design; (b) design of VSC with novel nonlinear switching surfaces is proposed for second order LTI systems; (c) step-by-step VSC and moving-surface VSC are proposed for VSC design of LTI systems, which can guarantee that the reaching condition of VSC is always met.

A class of single-input LTI systems is considered and described in Section 2. In Section 3, some definitions are provided, including sliding mode, SMD, RD, and URD, and then methods are developed for designing switching surfaces of linear systems whose state matrices are Hurwitz. In Section 4, the design of VSC for second order systems is investigated, where design of nonlinear switching surfaces is discussed. In Section 5, two design

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methods of VSC for high order linear systems are proposed. In Section 6, the proposed methods are used for attitude control of a flexible spacecraft to validate their effectiveness. Lastly, a concluding discussion is given in Section 7.

2. PROBLEM STATEMENT

The design of VSC for a single-input LTI system is considered

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u, \tag{1}$$

where the state vector $\boldsymbol{x}$ is n-dimensional, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are constant matrices of appropriate dimensions.

A linear switching surface is selected as

$$\sigma(x) = \boldsymbol{c}\boldsymbol{x} = 0, \tag{2}$$

where $\boldsymbol{c}$ is n-dimensional vector, $\boldsymbol{cB} = 1$, and (n-1) poles of the equivalent system are stable.

$$\dot{\boldsymbol{x}} = (\mathbf{I} - \boldsymbol{B}\boldsymbol{c})\boldsymbol{x}, \tag{3}$$

where $\mathbf{I}$ is a unity matrix.

The selection of the following control law

$$u = -\boldsymbol{c}\boldsymbol{A}x - Ksign(\sigma), \quad K > 0 \tag{4}$$

guarantees that the trajectory of the solution of the system described in (1) globally reaches onto (2) within finite time and is constrained on it. In this case, the closed-loop system is represented as the 'reaching law' [10]

$$\dot{\sigma} = -Ksign(\sigma), \tag{5}$$

satisfying the following 'reaching condition'

$$\dot{\sigma}\sigma < 0. \tag{6}$$

If the input of the system (1) is constrained by severe saturation, the control law (4) cannot be applied directly. However, studies on VSC from such a point of view are rare [7,8]. In this paper, we investigate the design of VSC for LTI systems with bounded inputs. That is, for the single-input LTI system (1) where the input is constrained in advance to

$$|u| \le K, \quad K > 0. \tag{7}$$

The following bounded control law is considered

$$u = -Ksign(\sigma). \tag{8}$$

3. DESIGN OF LINEAR SWITCHING SURFACES FOR LINEAR SYSTEMS

In this section, the SMD of VSS with linear switching surfaces is investigated. As a preliminary preparation, some definitions are provided as follows.

Definition 1 [3]**:** If, for any $\boldsymbol{x}_0$ on the switching surface $\sigma = 0$, we have $x(t)$ on $\sigma = 0$ for all $t > t_0$, then $x(t)$ is a sliding motion or sliding mode of the system.

Based on the above definition, the following three additional definitions are derived:

Definition 2: A domain D on the switching surface $\sigma = 0$ is a sliding mode domain (SMD) if every point on it undergoes the sliding motion.

Definition 3: A domain M in the state space is a reaching domain (RD) if the reaching condition (6) is satisfied in the domain.

Definition 4: A domain M in the state-space is an unescapable reaching domain (URD) if the motion starting from any initial state within M reaches onto SMD within a finite time T.

Remark 1: With the input constraint provided by equation (7), the SMD is not necessarily the whole switching surface, and it is often restricted to some local domain near zero in the state-space. Thus, the SMD is a subspace of the switching surface in which the reaching condition (6) is satisfied.

Remark 2: A point inside the URD does not necessarily satisfy reaching conditions, that is, the RD is a subspace of URD.

The closed-loop system (1) can be viewed as Lur'e-type system, i.e., a memory-less nonlinear feedback part (8) to the forward LTI system belongs to the sector [0 1). If the transfer function of the linear subsystem is so-called positive real, then it possesses important properties which may lead to the generation of a Lyapunov function for the whole system.

The following two lemmas describe a positive linear system and its stability.

Lemma 1 [11]**:** A minimum realization of the LTI system

$$G(s) = \boldsymbol{C}(s\mathbf{I} - \boldsymbol{AB})^{-1} + \boldsymbol{D} \tag{9}$$

is

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u, \\ y &= \boldsymbol{C}x + \boldsymbol{D}u. \end{aligned} \tag{10}$$

The above system is strictly passive, if $G(s)$ is strictly positive real.

Lemma 2 [11]**:** Considering a strictly passive system

$$\begin{aligned} \dot{\boldsymbol{x}} &= f(\boldsymbol{x}, u), \\ y &= h(\boldsymbol{x}, u), \end{aligned} \tag{11}$$

the origin of $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, u) = 0$ is globally asymptotically stable if a storage function of the system is radially unbounded.

Based on the Lemmas and Definitions presented above, global stabilization of the LTI system (1) by the bounded control (8) is considered. The following theorems are valid:

Theorem 1: For system (1), if $\boldsymbol{A}$ is Hurwitz and ($\boldsymbol{A}$, $\boldsymbol{B}$) is controllable, then by choosing the stable switching surface

$$\sigma(\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{x} = 0, \tag{12}$$

where $\boldsymbol{cB} = 1$ and ($\boldsymbol{c}$, $\boldsymbol{A}$) is observable, the SMD becomes

$$D_{SMD} := \{x \,|\, \boldsymbol{c}\boldsymbol{x} = 0, -K < \boldsymbol{c}\boldsymbol{A}\boldsymbol{x} < K\}, \tag{13}$$

and the URD is the whole state-space.

Proof: Firstly, concerning definition 2 and the linear switching surface (12), we have

$$\begin{aligned}\dot{\sigma} &= \boldsymbol{c}(\boldsymbol{Ax}+\boldsymbol{B}K) > 0 \quad \sigma < 0,\\ \dot{\sigma} &= \boldsymbol{c}(\boldsymbol{Ax}+\boldsymbol{B}K) < 0 \quad \sigma > 0.\end{aligned} \tag{14}$$

That is

$$-K < \boldsymbol{cAx} < K.$$

Therefore, the SMD is as provided in (13).

Secondly, the fact that the LTI system is a minimum realization of the strict positive transfer function (9) [12] combined with Lemma 1 indicates that the LTI system is strictly passive. A radial unbounded Lyapunov function can be chosen as a storage function by using KYP lemma as described in [11], and then from Lemma 2 it is guaranteed that the closed-loop system is globally exponentially stable. So if a sphere $\boldsymbol{N}$ near zero is considered such that

$$N(\boldsymbol{x},\gamma) := \{x \in R^n \mid \|\boldsymbol{x}\| \le \gamma\}, \tag{15}$$

where

$$0 < \gamma < \frac{K}{\|\boldsymbol{cA}\|}, \tag{16}$$

then the initial state from any point of the state-space reaches inside the sphere within a finite time. That is, the initial state from any point of the state-space approaches onto the SMD within a finite time.

Therefore, the control approach (8) guarantees that the trajectory of the solution starting from any initial state of (1) reaches onto the sliding mode domain on the switching surface within a finite time and approaches to zero thereafter.

Theorem 2: For system (1), if $\boldsymbol{A}$ is Hurwitz and $(\boldsymbol{A}, \boldsymbol{B})$ is controllable, then by choosing the stable switching surface (12) where $\boldsymbol{c}$ is

$$\boldsymbol{c} = \boldsymbol{a}[\boldsymbol{PM} - \rho\boldsymbol{P}]^{-1}, \tag{17}$$

and $\boldsymbol{a} = (a_0, a_1,\ldots,a_{n-1})$ are the coefficients of the characteristic polynomial of the system, $\boldsymbol{M}$ and $\boldsymbol{P}$ are

$$\boldsymbol{M} = \begin{pmatrix} 0 & \mathbf{I}_{n-1} \\ 0 & 0 \end{pmatrix}, \tag{18}$$

$$\boldsymbol{P} = \begin{pmatrix} 1 & & & \\ a_{n-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_1 & \cdots & a_{n-1} & 1 \end{pmatrix}, \tag{19}$$

respectively, and ρ is the solution of

$$\rho^n - a_{n-1}\rho^{n-1} + a_{n-2}\rho^{n-2} - \cdots + (-1)^n a_0 = 0.$$

If such $\boldsymbol{c}$ exists, the switching surface (12) is the SMD, and the URD is the whole state–space.

Proof: Firstly, it will be proven that the switching surface is SMD.

Concerning the transform matrix $\boldsymbol{P}$ (19), the controllable canonical form of (1) is given by

$$\dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}} + \bar{\boldsymbol{B}}u. \tag{20}$$

Then, $\bar{\boldsymbol{A}}$ $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{c}}$ are represented by

$$\bar{\boldsymbol{A}} = \boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix},$$

$$\bar{\boldsymbol{B}} = \boldsymbol{P}^{-1}\boldsymbol{B} = (0 \ \cdots \ 0 \ 1)^{\mathrm{T}},$$

$$\bar{\boldsymbol{c}} = (\bar{c}_1 \ \cdots \ \bar{c}_{n-1} \ 1)^{\mathrm{T}}.$$

The SMD (13) can be transformed as

$$-K < \sum_{i=1}^{n}\left(\bar{c}_{i-1} - a_{i-1}\right)x_i < K, \tag{21}$$

where $\bar{c}_0 = 0$.

Then, the SMD on the (n-1) dimensional switching surface is located between the parallel surfaces

$$\sum_{i=1}^{n}\left(\bar{c}_{i-1} - a_{i-1}\right)x_i = \pm K. \tag{22}$$

To guarantee that the SMD is the whole switching surface, the following conditions must be satisfied

$$\frac{-a_0}{\bar{c}_1} = \frac{\bar{c}_1 - a_0}{\bar{c}_2} = \cdots = \frac{\bar{c}_{n-2} - a_{n-2}}{\bar{c}_{n-1}} = \frac{\bar{c}_{n-1} - a_{n-1}}{1}. \tag{23}$$

That is,

$$(\bar{\boldsymbol{c}}\boldsymbol{M} - \boldsymbol{a}) = \rho\bar{\boldsymbol{c}}, \tag{24}$$

where ρ is a constant and $\boldsymbol{M}$ is given by (18).

Then, $\bar{\boldsymbol{c}}$ can be calculated from (24)

$$\bar{\boldsymbol{c}} = \boldsymbol{a}[\boldsymbol{M} - \rho\mathbf{I}]^{-1}, \tag{25}$$

where ρ is the solution of the following equation [10].

$$\rho^n - a_{n-1}\rho^{n-1} + a_{n-2}\rho^{n-2} - \cdots + (-1)^n a_0 = 0.$$

Then the representation of $\boldsymbol{c}$ in (17) can be derived from (25) and $\boldsymbol{P}$

$$\boldsymbol{c} = \bar{\boldsymbol{c}}\boldsymbol{P}^{-1} = \boldsymbol{a}[\boldsymbol{PM} - \rho\boldsymbol{P}]^{-1}. \tag{26}$$

Secondly, to show the fact that any points starting from the state–space reaches onto the switching surface within finite time, an initial state $\bar{\boldsymbol{x}}_0$ such that

$$\bar{\boldsymbol{x}}_0 \in R^n, \qquad \sigma(\bar{\boldsymbol{x}}_0) = \bar{\boldsymbol{c}}\bar{\boldsymbol{x}}_0 > 0, \tag{27}$$

is considered. Then the control law $u = -K$ is obtained from (8). Here, if the trajectory of the solution starting from $\boldsymbol{x}_0$ approaches to zero without reaching onto the switching surface (12), then

$$\sigma(\bar{\boldsymbol{x}}(\tau)) > 0, \quad 0 \le {}^{\forall}\tau < +\infty, \quad \bar{\boldsymbol{x}}(0) = \bar{\boldsymbol{x}}_0, \tag{28}$$

must be valid. The state equation of the system with the control law $u = -K$ is

$$\dot{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{A}}\bar{\boldsymbol{x}} - \bar{\boldsymbol{B}}K = \bar{\boldsymbol{A}}(\bar{\boldsymbol{x}} - \bar{\boldsymbol{A}}^{-1}\bar{\boldsymbol{B}}K). \tag{29}$$

Since $\boldsymbol{A}$ is Hurwitz, the equivalent points of the system are $\bar{\boldsymbol{x}}_e = \bar{\boldsymbol{A}}^{-1}\bar{\boldsymbol{B}}K$, thus

$$\sigma(\bar{x}_e) = \bar{c}\bar{\mathbf{A}}^{-1}\bar{\mathbf{B}}K = -\frac{c_1 K}{a_0} < 0, \tag{30}$$

and this contradicts equation (28), which implies that the trajectory intersects the switching surface at least once. Therefore, the trajectory of the system always reaches onto the switching surface (12) within a finite time. The same is true for the case in which $\sigma(0) < 0$.

Thus, the control approach (8) guarantees that the trajectory of the solution starting from any initial state of (1) reaches onto the sliding mode domain within a finite time and approaches to zero thereafter.

The following statements hold true concerning the above theorems:

Remark 3: In Theorem 1, the SMD is not the whole switching surface and is restricted to some local subspace of the switching surface. Therefore, RD is also restricted to some local domain near zero in the state-space. The SMD on the switching surface is maximized when c is chosen to minimize $\|\boldsymbol{cA}\|$. The larger the SMD, the larger is the robust region of the system.

Remark 4: In Theorem 2, the SMD is extended to the whole switching surface, that is, the robust region of the closed system is maximized, and the RD is $\{\boldsymbol{x} | -K \le \boldsymbol{cx} \le K\}$, which is a subspace of URD.

Remark 5: Generally speaking, the URD is not the whole state–space when $\boldsymbol{A}$ is not Hurwitz.

4. DESIGN OF VSC FOR SECOND ORDER LINEAR SYSTEMS WITH BOUNDED INPUTS

In this section, design of VSC with linear and nonlinear switching surfaces for the second order system described by (31) is investigated. The design of VSC is considered based on zeroing the output $y = x_1$ of the second order system given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -a_0 x_1 - a_1 x_2 + u, \end{aligned} \tag{31}$$

where the control input u is described by equation (8).

Concerning Theorem 2, if $a_1^2 - 4a_0 \ge 0$, the slope of the switching surface is given by

$$c_{1,2} = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}. \tag{32}$$

Then, the SMD and the RD are

$$D_{SMD} := \{(x_1, x_2) \mid cx_1 + x_2 = 0\}, \tag{33}$$

$$D_{RD} := \{(x_1, x_2) \mid |cx_1 + x_2| < K\}, \tag{34}$$

where c is shown in (32).

The URD is the whole state-space as shown in Fig. 1.

Concerning Theorem 1, if the parameter c is not chosen as described by (32), we obtain

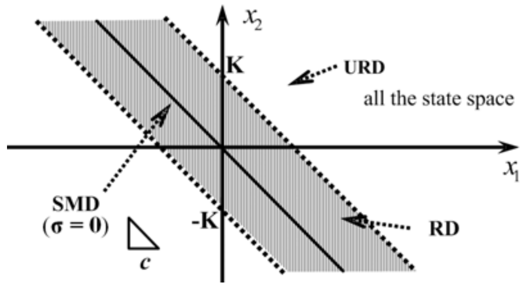


Fig. 1. Phase plane of SMD from Theorem 2.

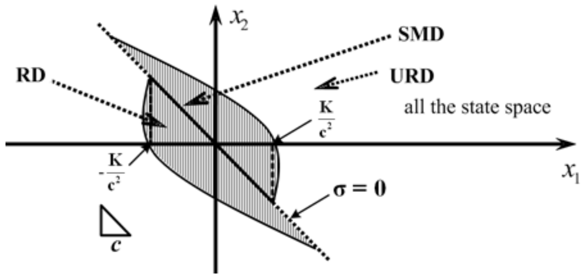


Fig. 2. Phase plane of SMD from Theorem 1.

$$c^2 - a_1 c + a_0 \ne 0. \tag{35}$$

Then the SMD of the system is given by

$$D_{SMD} := \left\{ (x_1, x_2) \mid cx_1 + x_2 = 0, |x_1| < \frac{K}{|c^2 - a_1 c + a_0|} \right\}, \tag{36}$$

and the URD is the whole state-space as shown in Fig. 2. The SMD on the switching surface is maximized when c is chosen to minimize $|c^2 - a_1 c + a_0|$. However, as can be seen from Fig. 2, the SMD will never be the whole switching surface.

As the subsequent step, the design of VSC with nonlinear switching surfaces for the second order system described by equation (31) is considered. In fact, nonlinear switching surfaces are not usually adopted in the literature due to their computational complexity, as compared to linear switching surfaces. However, it shows advantages in dealing with many cases, such as terminal sliding mode and input saturation [8,9].

The switching surface is chosen as follows [10]

$$\sigma(\boldsymbol{x}) = C(x_1) + x_2, \tag{37}$$

where $C(x_1)$ is a function of x_1.

The following theorem is said to be valid based on the proof that follows.

Theorem 3: For the system described by (31), the state trajectory on the sliding motion $C(x_1) + x_2 = 0$ is stable if $C(x_1)$ satisfies

$$C(x_1)x_1 > 0, \quad x_1 \ne 0. \tag{38}$$

Proof: Chose a Lyapunov function as follows

$$V(\boldsymbol{x}) = \frac{1}{2}x_1^2. \tag{39}$$

Thus,

Table 1. SMD of the second order system.

Parameter	G/C	SMD of variable structure control system
a_0=0 a_1=0	$C(x_1)$	$x_2 = -cx_1,\ \lvert x_1\rvert < \dfrac{k}{c^2}.$
	$G_1(x_1)$	$x_2 = -\text{sgn}(x_1)\sqrt{2g\lvert x_1\rvert}.$
	$G_2(x_1)$	$x_2 = -\text{sgn}(x_1)\sqrt{2g_1x_1\arctan(x_1) - g_1\log(1+x_1^2)}\ .$
a_0=0	$C(x_1)$	$x_2 = -cx_1,\ \lvert x_1\rvert < \dfrac{k}{c(c-a_1)}.$
	$G_1(x_1)$	$h^{-1}(x_1) + x_2 = 0,\quad h(x_1) = \dfrac{1}{a_1} - \dfrac{g}{a_1^2}\text{sgn}(x_1)\log\left(1+\dfrac{a_1}{g}\lvert x_1\rvert\right).$
a_1=0	$C(x_1)$	$x_2 = -cx_1,\ \lvert x_1\rvert < \dfrac{k}{c^2+a_0}.$
	$G_1(x_1)$	$a_0 > 0,\ x_2 = -\text{sgn}(x_1)\sqrt{-a_0x_1^2 + 2g\lvert x_1\rvert},\ \lvert x_1\rvert < \dfrac{2g}{a_0}.$ $a_0 < 0,\ x_2 = -\text{sgn}(x_1)\sqrt{-a_0x_1^2 + 2g\lvert x_1\rvert}.$
	$G_2(x_1)$	$a_0 > 0,$ No solution $a_0 < 0,\quad x_2 = -\text{sgn}(x_1)\sqrt{-a_0x_1^2 + 2g_1x_1\arctan(x_1) - g_1\log(1+x_1^2)}.$

$$\dot{V}(\boldsymbol{x}) = x_1\dot{x}_1 = -x_1C(x_1) < 0. \tag{40}$$

Then, the state trajectory on the sliding motion $C(x_1) + x_2 = 0$ is stable.

The first derivative of (37) is given by

$$\dot{\sigma}(\boldsymbol{x}) = G(x_1) - k\,\text{sgn}(s), \tag{41}$$

where

$$G(x_1) = -\frac{dC}{dx_1}C + (a_1C - a_0)x_1. \tag{42}$$

Then, the SMD is given by

$$D_{\text{SMD}} := \left\{\boldsymbol{x} \in \mathbb{R}^2 \mid C(x_1) + x_2 = 0,\ \lvert G(x_1)\rvert \le k\right\}. \tag{43}$$

The first step in the traditional design method is to choose $C(x_1)$. For example, when $C(x_1) = cx_1$, a linear switching surface (2) is obtained, and then the SMD and the RD can be calculated thereafter. It can be seen from (43) that if a function $G(x_1)$ that satisfies $|G(x_1)| \le k$ can be found, then the whole switching surface is SMD. The corresponding $C(x_1)$ can be derived from (42). Evidently, such $G(x_1)$ can be found easily.

Concerning the case where $a_0=a_1=0$, the following can be derived from (41)

$$\dot{\sigma}(\boldsymbol{x}) = -\frac{dC}{dx_1}C - k\,\text{sgn}(\sigma). \tag{44}$$

Assuming that

$$G_1(x_1) = g\,\text{sgn}(x_1),\quad 0 < g < k,$$

the following can be derived from (42)

$$\frac{dC}{dx_1}C = -g\,\text{sgn}(x_1). \tag{45}$$

A proper solution of (45) can be given as follows

$$C(x_1) = \text{sgn}(x_1)\sqrt{2g\lvert x_1\rvert}. \tag{46}$$

Then, the nonlinear switching surface is given by

$$\sigma_1(\boldsymbol{x}) = \text{sgn}(x_1)\sqrt{2g\lvert x_1\rvert} + x_2.$$

Besides, if we assume that

$$G_2(x_1) = -g_1\arctan(x_1),\quad 0 < g_1 < \frac{\sqrt{2}}{2}k,$$

another nonlinear switching surface can be derived as follows

$$\sigma_2(\boldsymbol{x}) = \text{sgn}(x_1)\sqrt{2g_1x_1\arctan(x_1) - g_1\log(1+x_1^2)} + x_2.$$

Examples of SMD of second order system are listed in Table 1, where different switching surfaces are chosen. It can be seen from Table1 that when a traditional linear switching surface is chosen, the SMD is not the whole linear switching surface, but rather it reduces to some local domain in the phase plane. However, the SMD can be the whole switching surface if a proper nonlinear switching function is chosen.

5. DESIGN OF VSC FOR HIGH ORDER LINEAR SYSTEMS WITH BOUNDED INPUTS

The controllable canonical form of (1) can be derived as follows by using transformation matrix $\boldsymbol{T}$

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}x_2, \\ \dot{x}_2 &= \boldsymbol{A}_{21}\boldsymbol{x}_1 + A_{22}x_2 + u, \end{aligned} \tag{47}$$

where the control input u is given by (8), and

$$\boldsymbol{T}\bar{\boldsymbol{A}}\boldsymbol{T}^{\mathrm{T}} = \boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & A_{12} \\ \boldsymbol{A}_{21} & A_{22} \end{bmatrix}, \quad \boldsymbol{T}\bar{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{0} \\ 1 \end{bmatrix}.$$

Without lose of generality, the objective of the controller is given by

$$\lim_{t\to\infty} \boldsymbol{x}(t) = 0.$$

It can be seen from Section 4 that any point in the URD, which does not satisfy the reaching condition may approach to zero within a finite time. However, it is difficult to find a general method for calculating the URD since it also depends on the structure of the system. As we know, any point in the RD will approach to zero within a finite time. Thus, if the initial state is in the RD, the stability of the closed-loop system can be guaranteed. The following discussions are based on this consideration.

5.1. Step-by-step variable structure controller

In this section, the RD and the SMD of the VSC will be investigated, and a novel design method of the VSC referred to as Step-by-Step Variable Structure Controller (SSVSC) will be proposed.

A linear switching surface is chosen as

$$\sigma = \boldsymbol{c}\boldsymbol{x} = \boldsymbol{c}_1\boldsymbol{x}_1 + x_2, \tag{48}$$

where $\boldsymbol{c} = [\boldsymbol{c}_1 \ 1]$.

By substituting (48) into (47), we obtain

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{A}_0\boldsymbol{x}_1 + \boldsymbol{B}_0\sigma, \tag{49}$$

$$\dot{\sigma} = \boldsymbol{C}_0\boldsymbol{x}_1 + D_0\sigma - k\,\mathrm{sgn}(\sigma), \tag{50}$$

where $\boldsymbol{A}_0 = \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{c}_1$, $\boldsymbol{B}_0 = \boldsymbol{A}_{12}$, $\boldsymbol{C}_0 = \boldsymbol{c}_1\boldsymbol{A}_{11} + \boldsymbol{A}_{21} - \boldsymbol{c}_1\boldsymbol{A}_{12}\boldsymbol{c}_1 - \boldsymbol{A}_{22}\boldsymbol{c}_1$, $D_0 = \boldsymbol{c}_1\boldsymbol{A}_{12} + \boldsymbol{A}_{22}$.

If $\sigma = 0$, equation (49) can be expressed as

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{A}_0\boldsymbol{x}_1, \tag{51}$$

which represents the sliding motion, where $\boldsymbol{A}_0 = \boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{c}_1$. And $\boldsymbol{c}_1$ can be designed by using the pole placement method.

Equation (50) represents a reaching motion, which can be used to investigate the RD. Assume that

$$\left[\left(\|\boldsymbol{C}_0\| + |D_0|\|\boldsymbol{c}_1\|\right)\|\boldsymbol{x}_1\| + |D_0||x_2|\right] < m, \tag{52}$$

then

$$\boldsymbol{C}_0\boldsymbol{x}_1 + D_0\sigma \le \|\boldsymbol{C}_0\|\|\boldsymbol{x}_1\| + |D_0||\sigma| < m.$$

From (50), we can obtain

$$\begin{aligned} \sigma\dot{\sigma} &= \sigma(\boldsymbol{C}_0\boldsymbol{x}_1 + D_0\sigma - k\,\mathrm{sgn}(\sigma) + d) \\ &< \sigma(m - k\,\mathrm{sgn}(\sigma) + d) < 0. \end{aligned}$$

That is, $\sigma\dot{\sigma} < 0$ is established.

It can be seen from the above analysis that if the control input is bounded, the RD is not the whole state–space. Rather, it reduces to some local domain that satisfies (52), as shown in Fig. 3. Therefore, the SSVSC method is proposed, which involves the following four steps:

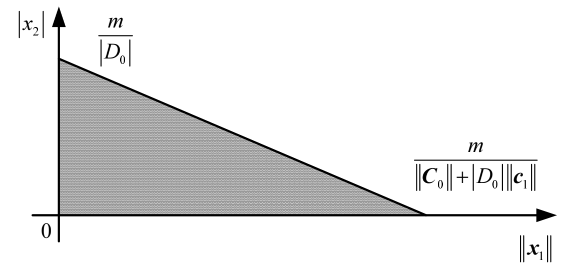


Fig. 3. SMD in phase plane.

Step 1: Choose $\boldsymbol{c}_1$ in (48) to stabilize the sliding motion in equation (51);

Step 2: Calculate the RD from equation (52): $\|\boldsymbol{x}\| < \alpha$, $\alpha > 0$;

Step 3: Divide the VSC process into n steps where $n = \min_m \left\{ m \mid m \ge \frac{\|\boldsymbol{x}(0)\|}{\alpha}, m \in \mathbb{Z} \right\}$, and $\boldsymbol{x}(0)$ is the initial state of the system;

Step 4: SSVSC strategy: The control law given by (7) is applied to the i th ($i < n$) step of the VSC whose initial state is $\left(n \times \frac{\boldsymbol{x}(0)}{\alpha} \right)$. When the control accuracy meets certain conditions (i.e., $\|\boldsymbol{x}\| < \varepsilon, \varepsilon > 0$), the ($i$ + 1)th VSC process will start when $i + 1 = n$.

It can be seen that the state trajectory is always in RD by using SSVSC strategy.

5.2. Moving-surface variable structure controller

In this section, a novel nonlinear switching surface is proposed as follows

$$\sigma = \boldsymbol{c}(\boldsymbol{x} - \boldsymbol{x}(0)e^{-\lambda t}) = \boldsymbol{c}_1\boldsymbol{x}_1 + x_2 - \boldsymbol{c}\boldsymbol{x}(0)e^{-\lambda t}, \ \lambda > 0, \tag{53}$$

where $\boldsymbol{c} = [\boldsymbol{c}_1 \ 1]$. Then, $\sigma(\boldsymbol{x}(0)) = 0$ stands for a cluster of parallel sliding surfaces with (48).

From (47) and (53), we obtain

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{A}_0\boldsymbol{x}_1 + \boldsymbol{B}_0\sigma + \boldsymbol{E}_0\boldsymbol{x}(0)e^{-\lambda t}, \tag{54}$$

$$\dot{\sigma} = \boldsymbol{C}_0\boldsymbol{x}_1 + D_0\sigma + F_0\boldsymbol{x}(0)e^{-\lambda t} - k\,\mathrm{sgn}(\sigma) + d, \tag{55}$$

where $\boldsymbol{E}_0 = \boldsymbol{A}_{12}\boldsymbol{c}$ and $F_0 = (A_{22} + \lambda + \boldsymbol{c}_1\boldsymbol{A}_{12})\boldsymbol{c}$.

It can be seen from (53) that $\boldsymbol{x}(0)$ is on the sliding surface. Assume that the control input (7) could make the state trajectory stay on the sliding surface. From (54), we obtain

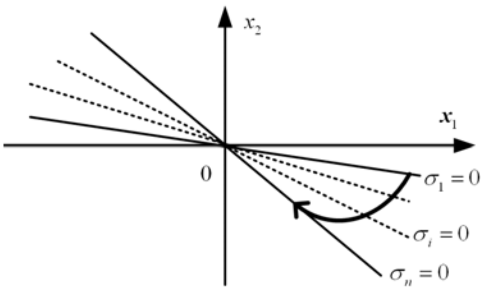
$$\dot{\boldsymbol{x}}_1 = \boldsymbol{A}_0\boldsymbol{x}_1 + \boldsymbol{E}_0\boldsymbol{x}(0)e^{-\lambda t}. \tag{56}$$

By integrating (56), we obtain

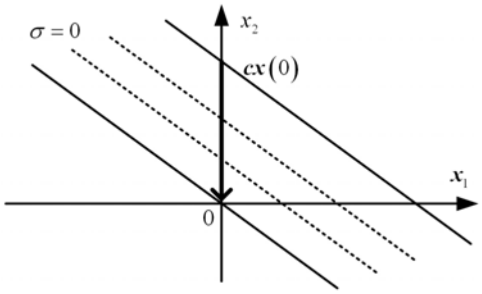
$$\boldsymbol{x}_1 = \int_0^t \boldsymbol{A}_0\boldsymbol{x}_1 dt - \frac{1}{\lambda}\boldsymbol{E}_0\boldsymbol{x}(0)e^{-\lambda t} + \boldsymbol{x}_1(0). \tag{57}$$

Since $\boldsymbol{A}_0$ is negative definite, then

$$\|\boldsymbol{x}_1\| \le 2\|\boldsymbol{x}_1(0)\| + \frac{\|\boldsymbol{E}_0\|}{\lambda}\|\boldsymbol{x}(0)\| \le \left(2 + \frac{\|\boldsymbol{E}_0\|}{\lambda}\right)\|\boldsymbol{x}(0)\|. \tag{58}$$



(a) Switching surfaces of SSVSC.



(b) Switching surfaces of MSVSC.

Fig. 4. Switching surfaces in phase plane.

From (58), we obtain

$$\boldsymbol{C}_0\boldsymbol{x}_1+F_0\boldsymbol{x}(0)e^{-\lambda t}\le\|\boldsymbol{C}_0\|\|\boldsymbol{x}_1\|+\|F_0\boldsymbol{x}(0)\|$$
$$\le\left[\|\boldsymbol{C}_0\|\left(2+\frac{\|\boldsymbol{E}_0\|}{\lambda}\right)+\|F_0\|\right]\|\boldsymbol{x}(0)\|.$$

From (55) and (58), if the following inequality is satisfied, then $\sigma\dot{\sigma}<0$ and

$$\left[\|\boldsymbol{C}_0\|\left(2+\frac{\|\boldsymbol{E}_0\|}{\lambda}\right)+\|F_0\|\right]<\frac{m}{\|\boldsymbol{x}(0)\|}. \tag{59}$$

It can be seen that the RD is independent of the initial state, and the state trajectory could stay on the sliding surface $\sigma=0$ if proper $\boldsymbol{c}$ and λ are chosen. Besides, it can be seen from (53) that the state trajectory goes to zero on $\boldsymbol{c}_1\boldsymbol{x}_1+x_2=0$ when $t\to\infty$.

The design process of Moving-Surface Variable Structure Controller (MSVSC) involves the following steps:

Step 1: Choose $\boldsymbol{c}_1$ in (48) to stabilize the sliding motion given by (51);

Step 2: Calculate λ by Substituting $\boldsymbol{c}_1$ into (59). If such λ does not exit, then go back to step 1.

The switching surfaces of the SSVSC and the MSVSC are shown in Fig. 4.

6. SIMULATIONS

In order to demonstrate the effectiveness of the proposed control schemes, numerical simulations are performed and presented in this section with an illustrative example pertaining to the attitude control of a flexible spacecraft. The dynamic equations of a flexible spacecraft undergoing maneuvering are given as follows

$$J\ddot{\theta}+\boldsymbol{\upsilon}^{\mathrm{T}}\ddot{\boldsymbol{\eta}}=u+d(t), \tag{60}$$
$$\ddot{\boldsymbol{\eta}}+\boldsymbol{C}\dot{\boldsymbol{\eta}}+\boldsymbol{K}\boldsymbol{\eta}+\boldsymbol{\upsilon}\ddot{\theta}=0, \tag{61}$$

where J is the moment of inertia of the flexible spacecraft, θ is the attitude angle, $\boldsymbol{\upsilon}\in\mathbb{R}^{N_\eta}$ is the coupling vector between attitude and vibration modes, $K=diag\{w_{ni}^2,i=1,\cdots,N_\eta\}\in\mathbb{R}^{N_\eta\times N_\eta}$ is the stiffness matrix, $\boldsymbol{C}=diag\{2\zeta_i w_{ni},i=1,\cdots,N_\eta\}\in\mathbb{R}^{N_\eta\times N_\eta}$ is the damping matrix, u is the control input acting on the rigid hub of the flexible spacecraft, and $\boldsymbol{\eta}\in\mathbb{R}^{N_\eta}$ is the modal coordinate vector. N_η is the number of flexible modes.

Let $\bar{\boldsymbol{x}}=[\theta\quad\boldsymbol{\eta}\quad\dot{\theta}\quad\dot{\boldsymbol{\eta}}]^{\mathrm{T}}$ and transform equations (60)-(61) into the state–space equation

$$\dot{\bar{\boldsymbol{x}}}=\bar{\boldsymbol{A}}\bar{\boldsymbol{x}}+\bar{\boldsymbol{B}}u, \tag{62}$$

where

$$\bar{\boldsymbol{A}}=\begin{bmatrix}\boldsymbol{0} & \boldsymbol{I}\\ -\boldsymbol{M}^{-1}\bar{\boldsymbol{K}} & -\boldsymbol{M}^{-1}\bar{\boldsymbol{C}}\end{bmatrix},\quad \bar{\boldsymbol{B}}=\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{M}^{-1}\boldsymbol{E}\end{bmatrix},$$
$$\boldsymbol{M}=\begin{bmatrix}J & \boldsymbol{\upsilon}^{\mathrm{T}}\\ \boldsymbol{\upsilon} & \boldsymbol{I}\end{bmatrix},\ \boldsymbol{E}=[1\quad\boldsymbol{0}]^{\mathrm{T}},\ \bar{\boldsymbol{K}}=\begin{bmatrix}0 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{K}\end{bmatrix},$$
$$\bar{\boldsymbol{C}}=\begin{bmatrix}0 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{C}\end{bmatrix}.$$

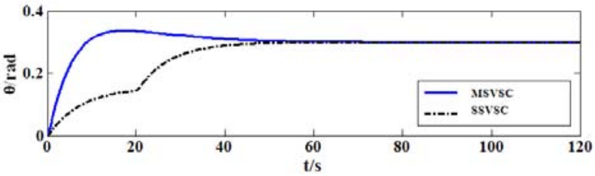
In this study, pointing control and simultaneous vibration suppression are the main objectives. Without loss of generality, it is assumed that the flexible spacecraft maneuvers from an initial state $\boldsymbol{x}(t_0)$ to $\boldsymbol{x}(t_f)$, that is

$$\lim_{t\to\infty}\boldsymbol{x}(t)=\boldsymbol{x}(t_f).$$

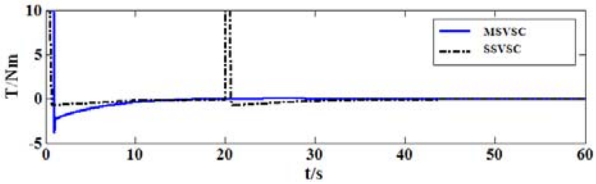
The parameters of the simulated flexible spacecraft are chosen as follows: $J=280\mathrm{kg\cdot m^2}$, $w=0.768$, $N_\eta=1$, $\boldsymbol{\upsilon}=[-1.256\ \ 0.918\ \ -1.672]\ \mathrm{kg^{1/2}m/s^2}$, ζ=0.0056, $|u|\le10$ Nm, $\theta(0)=\boldsymbol{\eta}(0)=0$, $\theta(t_f)=0.3$ rad, and $\boldsymbol{\eta}(t_f)=0$.

The poles of the linear system (51) are set to (−1 −1 −1), where $\boldsymbol{c}$ =[0.005 −0.457 −0.148 1.000]. The parameters of the SSVSC are $\alpha=0.1538$, and $\varepsilon=10^{-2}$rad. Thus, the attitude maneuvering can be divided into the following two steps: (0 0)→(0.15 0) →(0.3 0). The parameter of the MSVSC is λ= 0.1.

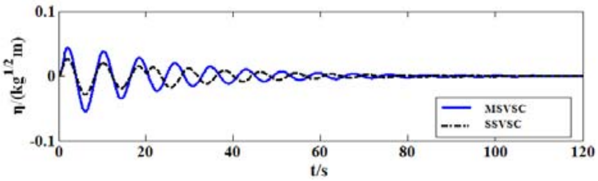
It can be seen from Fig. 5 that the desired angular displacement is accurately achieved with simultaneous vibration reduction. According to Figs. 5(a) and 5(b), the switching time of the SSVSC is at about 20 sec. The initial states are in the RD at each step, and its trajectory stays on the switching surfaces once they meet, as shown in Fig. 5(d). Compared with the SSVSC, the control input of the MSVSC experiences relatively long duration of action at the beginning of the maneuvering, which results in 13% overshoot in the attitude angle (see Fig. 5(a)) and large amplitude of the elastic coordinates (see Fig. 5(c)). Moreover, if the switching time is chosen as an odd multiple of half cycle of the elastic vibration, the elastic vibration will be reduced efficiently.



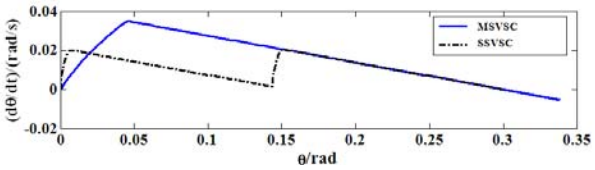
(a) Time response of attitude angle.



(b) Time response of control torque.



(c) Time response of modal coordinate.



(d) Phase plane of attitude angle.

Fig. 5. Simulation results of a flexible spacecraft.

7. CONCLUSIONS

In this paper, design of VSC for single-input LTI system with bounded inputs was studied. The concepts of SMD, RD and URD were introduced and it was found the URD is the whole phase plane when the state matrix of a linear system is Hurwitz and the SMD is reduced to some local domain in the linear switching surface when the state matrix of a linear system is not Hurwitz. For a second order system, the SMD is the whole switching surface if a proper nonlinear switching function is selected. Moreover, the SSVSC and MSVSC methods proposed in the paper can be used for designing the VSC for LTI systems with bounded inputs. Finally, the results of numerical simulations demonstrated the effectiveness of the proposed methods.

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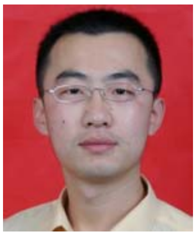

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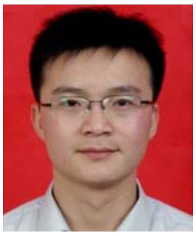

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